HIGHER-RANK NUMERICAL RANGES OF UNITARY AND NORMAL MATRICES

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Abstract. We verify a conjecture on the structure of higher-rank numerical ranges for a wide class of unitary and normal matrices. Using analytic and geometric techniques, we show precisely how the higher-rank numerical ranges for a generic unitary matrix are given by complex polygons determined by the spectral structure of the matrix. We discuss applications of the results to quantum error correction, specifically to the problem of identification and construction of codes for binary unitary noise models.

1. Introduction

The study of higher-rank numerical ranges of matrices was initiated in [**6**], with a basic problem in quantum error correction [**5**] giving the primary motivation. Higher-rank numerical ranges generalize the classical numerical range of a matrix, and arise as a special case of the matricial range for a matrix [**8, 15**]. In [**6**], three of us conjectured that the higher-rank numerical ranges of normal matrices depend, in a very precise way, on the spectral structure of the matrix. The conjecture reduces in the rank-1 case to a well-known property of the classical numerical range for a normal matrix, and it has opened the door to some interesting new mathematical problems. Its verification (or refutation) would also yield information for quantum error correction.

In this paper we verify the higher-rank numerical range conjecture for a wide variety of unitary and normal matrices. This is accomplished by introducing a number of new geometric techniques into the analysis. We show in the case of a generic $N \times N$ unitary with non-degenerate spectrum and positive integer $k \geq 1$ with $N \geq 3k$, that the kth numerical range is given by a certain polygon in the complex plane determined by the eigenvalues of the unitary, and thus verify the conjecture. In

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other cases, such as $N = 5m$ and $k = 2m$, we also verify the conjecture. But the analysis in these cases is more delicate, and our proof is non-constructive in nature. Figure 1 below provides a chart indicating the cases we verify, together with the various constraints.

Our results may be applied to construct error correcting codes for a special class of quantum channels. A "binary unitary channel" [**5**] $\mathcal{E}$ is a noise model described by two unitary errors that can occur during the time evolution of a given quantum system. The construction of codes for such a channel relies on the structure of the higher-rank numerical ranges for a single unitary U. Suppose U acts on N-dimensional Hilbert space. Then an "$[N, k]$-code" for $\mathcal{E}$ is a k-dimensional subspace code that is correctable for $\mathcal{E}$. The results described above for $N \geq 3k$ yield a simple algorithm to determine the existence of codes, and an explicit construction of codes when they exist.

The paper is organized as follows. Section 2 contains a discussion of some basics of quantum error correction that give important motivation for consideration of higher-rank numerical ranges. In Section 3 we discuss the conjecture, and show in particular how the general normal case relies on the unitary case. Section 4 deals with the structure of the aforementioned polygon and a derivation of conditions under which it is nonempty. Section 5 includes the proof for $N \geq 3k$ and the construction of codes for binary unitary channels. In Section 6 we derive a number of other cases non-constructively based on the case $N = 5$ and $k = 2$. We finish with a brief discussion on possible further avenues of research and limitations in Section 7.

2. Error Correction in Quantum Computing and Binary Unitary Channels

For a more complete discussion on the material of this section see [**5**] and the references therein. We start with a quantum system in contact with an external environment having finitely many degrees of freedom represented on a Hilbert space $\mathcal{H}$ such that $\dim \mathcal{H} = N < \infty$. Consider a unitary time evolution of the combined system and environment induced by a given Hamiltonian associated with some quantum computation implemented on $\mathcal{H}$. The action of the evolution map on the system is obtained by tracing out the environment, and the resulting map is called a quantum *channel* or *operation*. Such a map is described by a completely positive, trace preserving map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, and can always be represented in the operator-sum form as $\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$ for a set of operators $\{E_a\} \subseteq \mathcal{L}(\mathcal{H})$ satisfying $\sum_a E_a^\dagger E_a = I$. As a convenience we shall write $\mathcal{E} = \{E_a\}$ when the operators E_a determine $\mathcal{E}$ in this way. The E_a are interpreted as the noise or errors induced by $\mathcal{E}$.

In the standard approach to quantum error correction, a *code* on $\mathcal{H}$ is given by a subspace $\mathcal{C} \subseteq \mathcal{H}$ of dimension at least two. Denote the projection of $\mathcal{H}$ onto $\mathcal{C}$ by $P_\mathcal{C}$. A code $\mathcal{C}$ is (ideally) *correctable* for an operation $\mathcal{E}$ if there is a quantum operation $\mathcal{R} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ that acts as a left inverse of $\mathcal{E}$ on $\mathcal{C}$;

$$(1) \qquad \big(\mathcal{R} \circ \mathcal{E}\big)(\rho) = \rho \qquad \forall \rho \in P_\mathcal{C}\, \mathcal{L}(\mathcal{H})\, P_\mathcal{C}.$$

Given a representation for $\mathcal{E} = \{E_a\}$, a code $\mathcal{C}$ is correctable for $\mathcal{E}$ if and only if there are complex numbers (λ_{ab}) such that

$$(2) \qquad P_\mathcal{C} E_a^\dagger E_b P_\mathcal{C} = \lambda_{ab} P_\mathcal{C} \quad \forall\, a, b.$$

Thus, the problem of finding correctable codes for $\mathcal{E}$ is equivalent to simultaneously solving the family of equations in Eqs. (2) for the scalars λ_{ab} and projections $P_\mathcal{C}$, for all pairs a, b.

Of central importance in quantum computing, communication and cryptography is the class of *randomized unitary channels* [**1, 3**]. Such a channel has a representation of the form $\mathcal{E} = \{\sqrt{p_a}\, U_a\}$ where each U_a is a unitary operator and the $\{p_a\}$ form a classical probability distribution; $p_a > 0$, $\sum_a p_a = 1$. Hence,

$$(3) \qquad \mathcal{E}(\rho) = \sum_a p_a U_a \rho U_a^\dagger \qquad \forall \rho \in \mathcal{L}(\mathcal{H}).$$

The associated quantum operation is given by the scenario in which the error U_a occurs with probability p_a. By Eqs. (2), finding ideal correctable codes for $\mathcal{E} = \{\sqrt{p_a}\, U_a\}$ is equivalent to solving the (unnormalized) equations

$$(4) \qquad P U_a^\dagger U_b P = \lambda_{ab} P \quad \forall\, a, b,$$

for λ_{ab} and P. Note that each operator $U_a^\dagger U_b$ is unitary.

The following observation illustrates the importance of randomized unitary channels in error correction.

Proposition 2.1. *Let $\mathcal{E}$ be a quantum operation. Then $\mathcal{C}$ is a correctable code for $\mathcal{E}$ if and only if there is a randomized unitary channel $\mathcal{F}$ such that $\mathcal{C}$ is correctable for $\mathcal{F}$ and*

$$(5) \qquad \mathcal{E}(\rho) = \mathcal{F}(\rho) \qquad \forall \rho \in P_\mathcal{C}\, \mathcal{L}(\mathcal{H})\, P_\mathcal{C}.$$

Proof. In fact, $\mathcal{F} = \{\sqrt{p_a}\, U_a\}$ can be chosen so that the partial isometries $U_a P_\mathcal{C}$ have mutually orthogonal ranges for distinct a. This follows directly from the usual construction of a correction operation $\mathcal{R}$ for $\mathcal{E}$ on $\mathcal{C}$ [**10**]. The matrix (λ_{ab}) from Eqs.(2) is a density matrix, and the unitary which diagonalizes it can be used to find a set of error operators $\{F_a\}$ that implement $\mathcal{E} \circ \mathcal{P}_{\mathcal{C}}$, where $\mathcal{P}_{\mathcal{C}}(\cdot) = P_{\mathcal{C}}(\cdot)P_{\mathcal{C}}$, such that $P_{\mathcal{C}} F_a^\dagger F_b P_{\mathcal{C}} = \delta_{ab} d_{aa} P_{\mathcal{C}}$. The polar decomposition yields a partial isometry U_a, which can be extended to a unitary on the entire space, such that $F_a P_{\mathcal{C}} = U_a \sqrt{P_{\mathcal{C}} F_a^\dagger F_a P_{\mathcal{C}}} = \sqrt{d_{aa}}\, U_a P_{\mathcal{C}}$. The $\{\sqrt{d_{aa}}\}$ form a probability distribution by the trace preservation of $\mathcal{E}$. ■

Of course, the randomized unitary channel given by the restriction of $\mathcal{E}$ to the code $\mathcal{C}$ can only be obtained when the code itself is known. Thus, this result in itself is of little practical use for the general problem of finding error correcting codes. Nevertheless, it indicates that the general problem is equivalent to finding codes for the class of randomized unitary channels.

Observe that a code is correctable for a binary unitary noise model $\mathcal{E} = \{\sqrt{p}\, V, \sqrt{1-p}\, W\}$, where V and W are unitary, if and only if it is correctable for $\mathcal{E} = \{\sqrt{p}\, I, \sqrt{1-p}\, V^\dagger W\}$.

Definition 2.2. *A* **binary unitary channel** *on a Hilbert space $\mathcal{H}$ is a channel of the form*

$$\mathcal{E} = \{\sqrt{p}\, I, \sqrt{1-p}\, U\} \tag{6}$$

for some unitary $U \in \mathcal{L}(\mathcal{H})$ and fixed probability $0 < p < 1$. Thus, the action of $\mathcal{E}$ is given by

$$\mathcal{E}(\rho) = p\, \rho + (1-p)\, U\rho U^\dagger \qquad \forall \rho \in \mathcal{L}(\mathcal{H}). \tag{7}$$

Remark 2.3. *Binary unitary channels form a rather restrictive class of physical noise maps, but they provide a useful set of "toy" examples for testing the "compression" approach* [**5**] *to build quantum error correcting codes enabled by consideration of higher-rank numerical ranges. Observe that from Eqs. (2), the problem of finding ideal correctable codes for a given binary unitary channel is equivalent to solving four equations, but that the entire problem reduces to solving the single (unnormalized) equation for λ and P given by:*

$$PUP = \lambda\, P. \tag{8}$$

An immediate consequence of what follows is an algorithm to construct codes for a wide class of binary unitary channels. This is encapsulated in the discussion of Section 5.

3. Higher-Rank Numerical Range Conjecture

Given a fixed positive integer $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, the *kth numerical range* of T is the set of complex numbers

$$\Lambda_k(T) = \big\{\lambda \in \mathbb{C} : PTP = \lambda P \text{ for some rank}-k \text{ projection } P\big\}.$$

The classical numerical range $W(T) = \Lambda_1(T)$ is obtained when $k = 1$.

Definition 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $k \geq 1$ be a fixed positive integer. Then we define $\Omega_k(T)$ to be the intersection of the convex hulls* $\mathrm{conv}(\Gamma)$*, where Γ runs through all $(N-k+1)$-point subsets (counting multiplicities) of the set of eigenvalues* $\mathrm{spec}(T)$ *for T. That is,*

$$\Omega_k(T) \;\;=\;\; \bigcap_{\Gamma \subseteq \mathrm{spec}(T);\ |\Gamma| = N-k+1} \mathrm{conv}(\Gamma).$$

Thus, $\Omega_k(T)$ is a convex subset of the complex plane that can be computed directly from the spectrum of T. Below we will show how this set can typically be computed as an intersection of much fewer than $\binom{N}{N-k+1}$ sets.

It is easy to see that $\Omega_k(T)$ contains $\Lambda_k(T)$ for normal T. We include a short proof for completeness and notational purposes.

Proposition 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator and fix a positive integer $k \geq 1$. Then $\Lambda_k(T) \subseteq \Omega_k(T)$.*

Proof. Let $\{|\psi_1\rangle, \ldots, |\psi_N\rangle\}$ be a complete set of orthonormal eigenvectors for T with eigenvalues $T|\psi_j\rangle = \lambda_j|\psi_j\rangle$. Let $\lambda \in \Lambda_k(T)$ and let $P = \sum_{i=1}^k |\phi_i\rangle\langle\phi_i|$ be a rank-k projection such that $PTP = \lambda P$. Then $\langle T\phi_i|\phi_j\rangle = \delta_{ij}\lambda$ for all i, j. Let $\mathbb{A}$ be a subset of $\{1, \ldots, N\}$ with cardinality $|\mathbb{A}| = k-1$. Choose a unit vector $|\phi\rangle$ in the k-dimensional subspace $P\mathcal{H} = \mathrm{span}\{|\phi_1\rangle, \ldots, |\phi_k\rangle\}$ that is perpendicular to all the $|\psi_j\rangle$ for which $j \in \mathbb{A}$; and so, $|\phi\rangle = \sum_{j\notin\mathbb{A}} z_j|\psi_j\rangle$ with $\sum_j |z_j|^2 = 1$. Then we have

$$\lambda = \langle T\phi|\phi\rangle = \sum_{j\notin\mathbb{A}} |z_j|^2\lambda_j \;\;\in\;\; \mathrm{conv}\{\lambda_j : j \notin \mathbb{A}\}, \tag{9}$$

and it follows that λ belongs to $\Omega_k(T)$. ■

It is well-known and easy to verify that the numerical range of a normal operator T coincides with the convex hull of its eigenvalues (that is, $\Lambda_1(T) = \Omega_1(T)$). In **[6]** the following conjecture was asserted as a generalization of this fact.

Conjecture A. Let $\mathcal{H}$ be an N-dimensional Hilbert space and let $k \geq 1$ be a positive integer. Then for every normal operator $T \in \mathcal{L}(\mathcal{H})$,

$$\Lambda_k(T) \; = \; \Omega_k(T). \tag{10}$$

The Hermitian case [**6**] and the normal $N \leq 4$ case [**5**] of the conjecture have been verified previously. We show that the general normal case of Conjecture A can be reduced to the unitary case.

Proposition 3.3. *Conjecture A holds if and only if the conjecture holds for all unitary matrices.*

Proof. First note that for a fixed k, a standard translation argument shows the statement $\Lambda_k(T) = \Omega_k(T)$ for all normal $T \in \mathcal{L}(\mathbb{C}^N)$ is equivalent to the statement $0 \in \Lambda_k(T)$ if and only if $0 \in \Omega_k(T)$ for all normal $T \in \mathcal{L}(\mathbb{C}^N)$. We focus on the latter formulation.

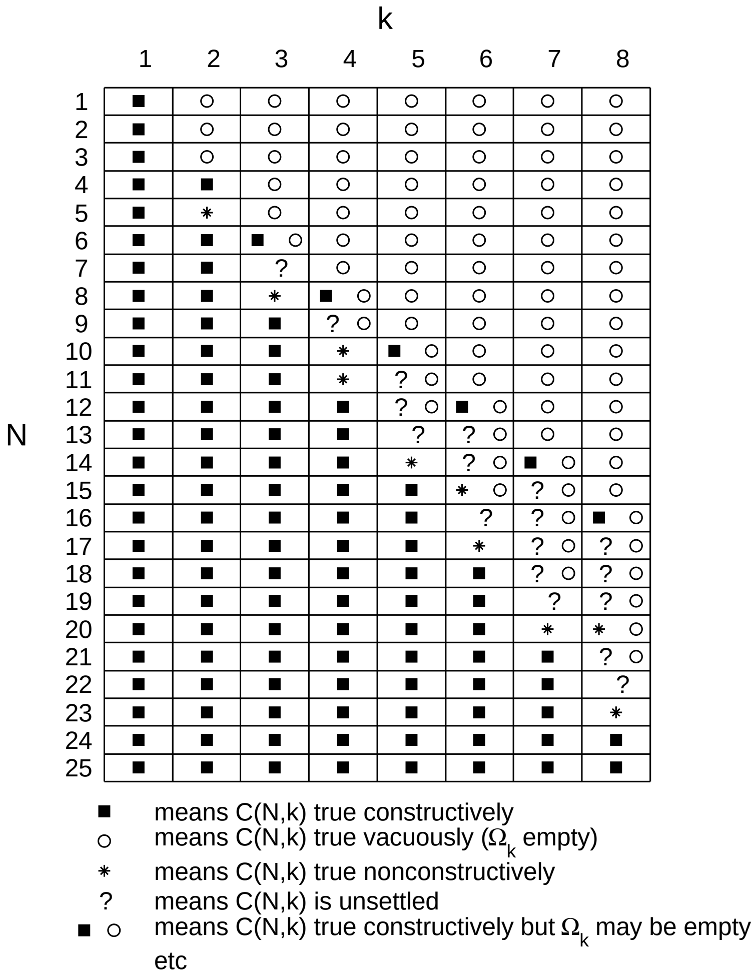
Every normal operator T decomposes as $T = T_1 \oplus 0_m$, where T_1 is normal and invertible. The case $m \geq k$ is easily handled, so assume $m < k$. One can check that $0 \in \Lambda_k(T) \quad$ if and only if $\quad 0 \in \Lambda_{k-m}(T_1)$, and $0 \in \Omega_k(T) \quad$ if and only if $\quad 0 \in \Omega_{k-m}(T_1)$. Let $\{\lambda_1, \ldots, \lambda_{N-m}\}$ be the (non-zero) eigenvalues for T_1, and let U be the unitary on $\mathbb{C}^{N-m}$ obtained from the polar decomposition of T_1 with eigenvalues $\{\frac{\lambda_1}{|\lambda_1|}, \ldots, \frac{\lambda_{N-m}}{|\lambda_{N-m}|}\}$. By assumption we have $\Lambda_{k-m}(U) = \Omega_{k-m}(U)$, and hence zero belongs to both sets or neither set. Thus, we complete the proof by showing that: (i) $0 \in \Lambda_{k-m}(T_1) \quad$ if and only if $\quad 0 \in \Lambda_{k-m}(U)$, and (ii) $0 \in \Omega_{k-m}(U) \quad$ if and only if $\quad 0 \in \Omega_{k-m}(T_1)$.

Note that by invertibility we have $T_1 = UR = RU = \sqrt{R}U\sqrt{R}$ where $R = \sqrt{T^\dagger T} \geq 0$ is invertible with eigenvalues $\{|\lambda_1|, \ldots, |\lambda_{k-m}|\}$. Thus, (i) follows from the more general principle that if $T = X^\dagger S X$ where X is invertible, then $0 \in \Lambda_k(T)$ if and only if $0 \in \Lambda_k(S)$. Indeed, if P is a rank-k projection such that $PTP = 0$, then the rank-k range projection Q of XP satisfies $QSQ = 0$.

For (ii), note that by definition $0 \notin \Omega_{k-m}(T_1)$ precisely when 0 does not belong to the convex hull of $N - k + m + 1$ of the eigenvalues $\{\lambda_1, \ldots, \lambda_{N-m}\}$. This is equivalent to the existence of a line passing through the origin that does not meet this convex hull. By the same argument, this geometric condition is equivalent to $0 \notin \Omega_{k-m}(U)$. ■

This result, combined with the motivation from quantum computing discussed above, naturally leads to a focus on the unitary case of the conjecture. We introduce the following nomenclature to delineate the generic unitary subcases.

Definition 3.4. *We will use the notation* $\mathrm{Conj}\,(N,k)$ *to denote the sub-conjecture of Conjecture A given by the statement Eq. (10) for a given pair* $[N,k]$ *and all unitary operators on* N*-dimensional Hilbert space with non-degenerate spectrum. Further, we will say* $\mathrm{Conj}\,(N,k)$ *is constructively verified for a given pair* $[N,k]$ *if it is shown that Eq. (10) holds for every unitary operator* U *on* $\mathcal{H}=\mathbb{C}^N$*, and if, whenever* $\Lambda_k(U)$ *is nonempty, for every* $\lambda\in\Lambda_k(U)$ *a rank-*k *projection* P *can be explicitly constructed such that* $PUP=\lambda P$*.*



N \ k	1	2	3	4	5	6	7	8
1	■	○	○	○	○	○	○	○
2	■	○	○	○	○	○	○	○
3	■	○	○	○	○	○	○	○
4	■	■	○	○	○	○	○	○
5	■	∗	○	○	○	○	○	○
6	■	■	■ ○	○	○	○	○	○
7	■	■	?	○	○	○	○	○
8	■	■	∗	■ ○	○	○	○	○
9	■	■	■	? ○	○	○	○	○
10	■	■	■	∗	■ ○	○	○	○
11	■	■	■	∗	? ○	○	○	○
12	■	■	■	■	? ○	■ ○	○	○
13	■	■	■	■	?	? ○	○	○
14	■	■	■	■	∗	? ○	■ ○	○
15	■	■	■	■	■	∗ ○	? ○	○
16	■	■	■	■	■	?	? ○	■ ○
17	■	■	■	■	■	∗	? ○	? ○
18	■	■	■	■	■	■	? ○	? ○
19	■	■	■	■	■	■	?	? ○
20	■	■	■	■	■	■	∗	∗ ○
21	■	■	■	■	■	■	■	? ○
22	■	■	■	■	■	■	■	?
23	■	■	■	■	■	■	■	∗
24	■	■	■	■	■	■	■	■
25	■	■	■	■	■	■	■	■

■ means C(N,k) true constructively
○ means C(N,k) true vacuously (Ω_k empty)
∗ means C(N,k) true nonconstructively
? means C(N,k) is unsettled
■ ○ means C(N,k) true constructively but Ω_k may be empty
etc

Figure 1. $\mathrm{Conj}(N,k)$ for nondegenerate unitary U.

4. The Structure of Ω_k

In this section we analyse the geometric structure of the set $\Omega_k(U)$ for a generic unitary U on N-dimensional Hilbert space with non-degenerate spectrum. Let us first establish notation we will use for the rest of the paper.

We shall consider the case of a unitary U with eigenvalues $\lambda_j = \exp(i\theta_j)$, $j = 1, \ldots, N$, such that $0 \le \theta_1 < \theta_2 < \ldots < \theta_N < 2\pi$. Thus, the eigenvalues λ_j are ordered counterclockwise around the unit circle $\partial\mathbb{D}$ in $\mathbb{C}$ (we use $\mathbb{D}$ to denote the closed unit disc). For multiple eigenvalues the numbering is arbitrary, but we choose an orthonormal system of eigenvectors $|\psi_j\rangle \in \mathcal{H}$ such that

$$U|\psi_j\rangle = \lambda_j|\psi_j\rangle. \tag{11}$$

When appropriate we extend the numbering of the λ_j and $|\psi_j\rangle$ cyclically: for example, λ_{N+1} means λ_1. Given integers i, j with $i < j \le i + N$, let $D(i, j, U)$ denote the compact convex subset of $\mathbb{C}$ bounded by the line segment from λ_i to λ_j and the counterclockwise circular arc from λ_j to λ_i; recall our conventions about cyclical numbering of the λ_j. We interpret $D(i, i + N, U)$ as $\{\lambda_i\}$.

We first show that the form of $\Omega_k(U)$ is simpler than what Definition 3.1 suggests. In particular, $\Omega_k(U)$ is a filled convex polygon with at most N sides.

Lemma 4.1. *For all $k \ge 1$ and every unitary $U \in \mathcal{L}(\mathbb{C}^N)$, the set $\Omega_k(U)$ is the convex polygon given by*

$$\Omega_k(U) = \bigcap_{i=1}^{N} D(i, i + k, U). \tag{12}$$

Proof. Let Ω denote the intersection of Eq. (12). The cardinality of the set

$$S = \{1, 2, \ldots, N\} \setminus \{i + 1, i + 2, \ldots, i + k - 1\}$$

(where integers are interpreted modulo N) is $|S| = N - k + 1$. Thus

$$\Omega_k(U) \subseteq \mathrm{conv}(\{\lambda_j : j \in S\}) \subseteq D(i, i + k, U).$$

Since this holds for all i, we have $\Omega_k(U) \subseteq \Omega$.

On the other hand, if $|S| = N - k + 1 = s$ we may write $S = \{i_1, i_2, \ldots, i_s\}$ with $1 \le i_1 < i_2 < \cdots < i_s \le N$ and

$$\mathrm{conv}(\{\lambda_j : j \in S\}) = \bigcap_{j=1}^{s} D(i_j, i_{j+1}, U)$$

(it is understood here that $i_{s+1} = i_1$). Since between i_j and i_{j+1} there are $i_{j+1} - i_j - 1$ integers that are omitted from S, we must have $i_{j+1} - i_j \leq k$, so that $D(i_j, i_{j+1}, U) \supseteq D(i_j, i_j + k, U)$. It follows that Ω is contained in $\mathrm{conv}(\{\lambda_j : j \in S\})$ for each such S. Hence $\Omega_k(U) \supseteq \Omega$, and equality is verified. ■

The following containments are direct consequences of this result.

Corollary 4.2. *For all U and all k, we have $\Omega_{k+1}(U) \subseteq \Omega_k(U)$.*

Proof. In view of Lemma 4.1, we need only observe that, for any i, $D(i, i+k+1, U) \subseteq D(i, i+k, U)$. ■

Corollary 4.3. *If V is unitary and $\mathrm{spec}(U) \subseteq \mathrm{spec}(V)$, then $\Omega_k(U) \subseteq \Omega_k(V)$ for all k.*

Proof. It is enough to treat the case where $\mathrm{spec}(V) = \mathrm{spec}(U) \cup \{\lambda_{N+1}\}$ and to arrange the geometric ordering so that $\mathrm{spec}(U) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ and $\mathrm{spec}(V) = \{\lambda_1, \lambda_2, \ldots, \lambda_{N+1}\}$. For $1 \leq i \leq N - k$ we have $D(i, i+k, V) = D(i, i+k, U)$; for $N - k < i \leq N$, we have $D(i, i+k, V) \supseteq D(i, i+k, U)$ (note that the extended numbering within $\mathrm{spec}(V)$ is done modulo $N+1$); finally, $D(N+1, N+1+k, V) \supseteq D(N, N+k, U)$. Using Lemma 4.1, we see that $\Omega_k(V) \supseteq \Omega_k(U)$. ■

The following corollary points out that $\mathrm{Conj}(N, k)$ is easy to verify when k divides N.

Corollary 4.4. *Suppose that k divides N, with $m = N/k$. Then $\Omega_k(U)$ is the intersection of k m–gons, and Conj(N, k) follows.*

Proof. Let $S_i = \{i, i+k, i+2k, \ldots, i+(m-1)k\}$ $(i = 1, 2, \ldots, k)$; these index sets partition $\{1, 2, \ldots, N\}$. In view of Lemma 4.1, the m–gons

$$G_i = \bigcap_{j \in S_i} D(i, i+k, U)$$

intersect to form $\Omega_k(U)$. Now consider $\lambda \in \Omega_k(U)$. Since $\lambda \in G_i$ for each i we may write

$$\lambda = \sum_{j \in S_i} t_{ij} \lambda_j$$

as a convex combination ($t_{ij} \geq 0$, $\sum_{j \in S_i} t_{ij} = 1$). For $i = 1, 2, \ldots, k$, let

$$|\phi_i\rangle = \sum_{j \in S_i} \sqrt{t_{ij}} |\psi_j\rangle;$$

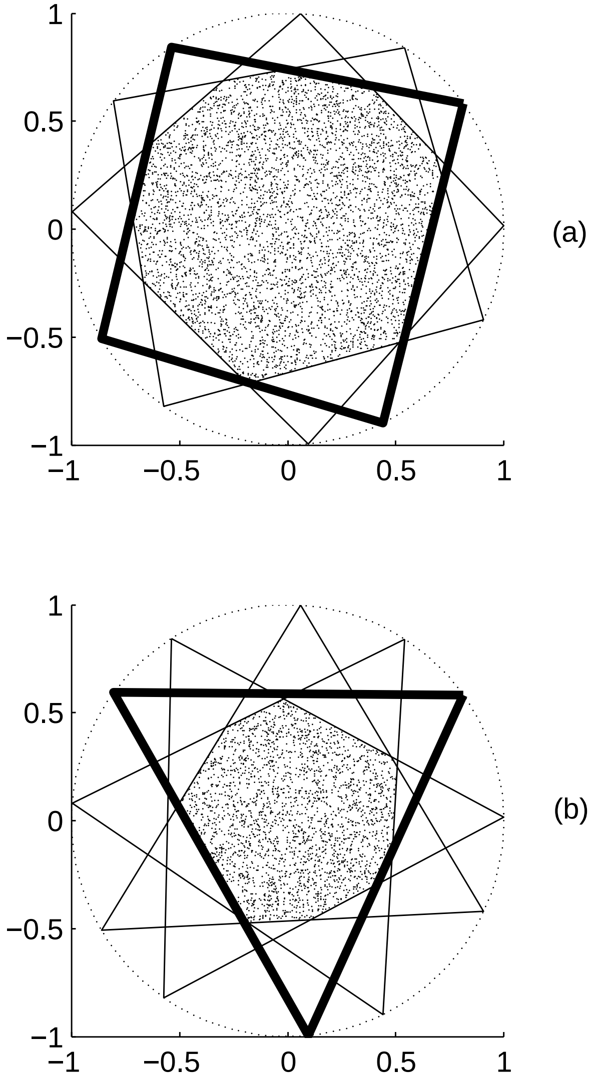


Figure 2. Corollary 4.4 in action: (a) $\Omega_3(U)$ as intersection of 3 quadrilaterals when $N = 12$; (b) $\Omega_4(U)$ as intersection of 4 triangles when $N = 12$.

clearly the $|\phi_i\rangle$ are unit vectors, and they are orthogonal because the S_i are disjoint. We see that $(U-\lambda I)|\phi_i\rangle \perp |\phi_{i'}\rangle$ for all i, i', so that $\mathcal{V} = \mathrm{span}\{|\phi_i\rangle : i = 1, 2, \ldots, k\}$ satisfies $(U-\lambda I)\mathcal{V} \perp \mathcal{V}$. We can thus define the rank-k projection $P = \sum_{i=1}^{k} |\phi_i\rangle\langle\phi_i|$ onto $\mathcal{V}$. Then $P(U-\lambda I)P = 0$ and it follows that $\lambda \in \Lambda_k(U)$. Together with Proposition 3.2, we have verified $\mathrm{Conj}(N, k)$ in these cases. ■

Remark 4.5. *The corollary above is usually of interest when $m \geq 3$. If $N = k$ ($m = 1$), then $\Omega_k(U)$ will be empty unless U is a scalar. If $N = 2k$ ($m = 2$), then $D(i, i+k, U) \cap D(i+k, i+N, U)$ is the line segment $[\lambda_i, \lambda_{i+k}]$. Thus $\Omega_k(U)$ is usually empty in this case for $k > 2$; in cases where* $\operatorname{spec}(U)$ *has a special symmetry, $\Omega_k(U)$ may be a single point.*

Under certain conditions Corollary 4.3 may be made more precise.

Proposition 4.6. *Suppose V is unitary on $N+1$-dimensional Hilbert space and* $\operatorname{spec}(V) = \{\lambda_1, \lambda_2, \ldots, \lambda_{N+1}\}$ *with distinct λ_j. If $\Omega_{k+1}(V) \neq \emptyset$, then*

$$\Omega_k(V) = \bigcup_{j=1}^{N+1} \Omega_k(V_j), \tag{13}$$

where V_j is the unitary with spectrum $\operatorname{spec}(V) \setminus \{\lambda_j\}$.

Proof. Arguing as in the proof of Corollary 4.3 we see that $\Omega_k(V_{N+1}) = \Omega_k(V) \cap C(N+1)$, where

$$C(N+1) = \bigcap_{j=N+1-k}^{N} D(j, j+k+1, V).$$

Let λ be a point in $\Omega_{k+1}(V)$ (which is nonempty by hypothesis). Then $\lambda \in C(N+1)$ by Lemma 4.1. Note also that $C(N+1)$ includes the counterclockwise arc $A(N+1)$ of $\partial\mathbb{D}$ from $\lambda_{N+k+1} = \lambda_k$ to λ_{N+1-k}; since $N+1-k > k$ (otherwise we would have $N+1 \leq 2k < 2(k+1)$ so that $\Omega_{k+1}(V)$ would be empty), the arc $A(N+1)$ has nonempty interior (relative to $\partial\mathbb{D}$). Likewise, for each $j = 1, 2, \ldots, N+1$ we have $\Omega_k(V_j) = \Omega_k(V) \cap C(j)$ where $C(j)$ is a convex set containing λ and an arc $A(j)$. It remains to show that $\bigcup_j C(j) = \mathbb{D}$. The arcs $A(j)$ (overlapping in general) cover all of $\partial\mathbb{D}$ and each $C(j)$ includes the "sector" $\operatorname{conv}(\{\lambda\} \cup A(j))$; these sectors certainly cover $\mathbb{D}$. ■

The previous result raises the general question of which $\Omega_k(U)$ are nonempty. This question is of course important for the identification and construction of error-correcting codes. In particular, $\operatorname{Conj}(N, k)$ may include cases where both $\Lambda_k(U)$ and $\Omega_k(U)$ are empty, but this is not helpful for the applications. The question is somewhat clarified by the following.

Theorem 4.7. *Let $U \in \mathcal{L}(\mathbb{C}^N)$ be a unitary with N distinct eigenvalues and let $k \geq 1$. Then we have the following conditions on $\Omega_k(U)$.*

(1) If $N < 2k$, then $\Omega_k(U) = \emptyset$.

(2) If $N = 2k$, then $\Omega_k(U)$ is empty if the line segments $[\lambda_j, \lambda_{j+k}]$ do not intersect, and, otherwise, it is the singleton set given by the intersection point of these line segments.

(3) If $2k < N < 3k - 2$, then $\Omega_k(U)$ can be either empty or nonempty.

For any unitary $U \in \mathcal{L}(\mathbb{C}^N)$, whether or not the eigenvalues are distinct, we have:

(4) If $N \geq 3k - 2$, then $\Omega_k(U)$ is always nonempty.

Proof. To see (1) note that $D(j, j+k, U) \cap D(j+k, j+2k, U)$ is the singleton $\{\lambda_{j+k}\}$ and that the $\{\lambda_{j+k}\}$ are distinct. See also Proposition 1 from [**5**]. The case (2) is a special case of Corollary 4.4. Here, $\Omega_k(U)$ is non-empty (and a singleton set) precisely when the line segments $[\lambda_j, \lambda_{j+k}]$ intersect in a common point.

In view of Corollary 4.3, it is sufficient for the statement (3) to provide an example of U with $N = 3(k-1)$ and $\Omega_k(U) = \emptyset$. Such an example is provided by grouping $k - 1$ of the λ_j close to and on the counterclockwise side of each of the cube roots of unity. It is then clear that

$$D(k-1, 2k-1, U) \cap D(2k-2, 3k-2, U) \cap D(3k-3, 4k-3, U) = \emptyset;$$

note that here $N = 3k - 3$ so that $3k - 2 \equiv 1$ and $4k - 3 \equiv k$. Again recalling Lemma 4.1, we see that *a fortiori* $\Omega_k(U) = \emptyset$.

Concerning the statement (4), we recall Helly's Theorem from convex analysis (see, for example, Chapter 3 of [**13**]): a family of compact, convex subsets of $\mathbb{R}^d$ has nonempty intersection provided any subfamily of size $d + 1$ has nonempty intersection. Since each $D(i, i + k, U)$ is compact and convex in $\mathbb{R}^2 \equiv \mathbb{C}$ we need only prove that $N \geq 3k - 2$ implies

$$D(a, a + k, U) \cap D(b, b + k, U) \cap D(c, c + k, U) \neq \emptyset,$$

then invoke Helly's Theorem for $d = 2$ and Lemma 4.1. Such a triple intersection could only be empty if the complements in $\mathbb{D}$ covered all of $\mathbb{D}$. In particular, the omitted arcs (strictly between λ_a and λ_{a+k}, etc) would cover $\partial\mathbb{D}$. Each of these arcs contains $k-1$ points from spec(U), so we would have the contradiction $N \leq 3(k - 1)$. ■

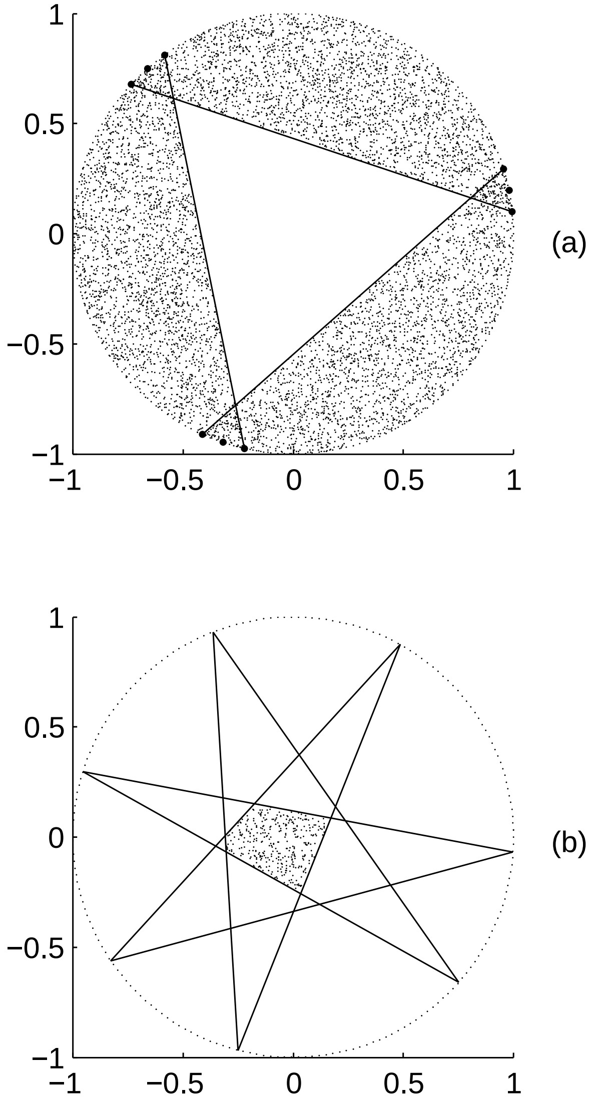


Figure 3. Illustrating Theorem 4.7: (a) Here $N = 9$ and $k = 4$; since $N < 3k - 2$ it may happen, as here, that $\Omega_4(U) = \emptyset$; note that $D(3, 7, U)$, $D(6, 1, U)$, and $D(9, 4, U)$ overlap pairwise but have no common point. (b) Here $N = 7$ and $k = 3$; since $N \geq 3k - 2$, the core $\Omega_3(U)$ of this 7–pointed star must be nonempty.

5. Verification of the Conjecture for $N \geq 3k$ and Construction of Codes for Binary Unitary Channels

In this section we give a constructive verification of $\mathrm{Conj}(N, k)$ in the cases $N \geq 3k$. In such cases we explicitly construct error-correcting codes. We require more notation.

Let $\Delta_k(U)$ denote the set of those $\lambda \in \mathbb{C}$ such that for some k *disjoint* subsets $S_1, S_2, \ldots, S_k$ of $\{1, 2, \ldots, N\}$ we have $\lambda \in \mathrm{conv}(\{\lambda_j : j \in S_i\})$ for each i. The Δ in the notation $\Delta_k(U)$ is to recall the "disjoint" condition. Evidently $\Delta_k(U) \subseteq \Delta_k(V)$ whenever $\mathrm{spec}(U) \subseteq \mathrm{spec}(V)$.

A straightforward generalization of the construction used in Corollary 4.4 yields the following result.

Lemma 5.1. *For every unitary $U \in \mathcal{L}(\mathbb{C}^N)$ and $k \geq 1$ we have $\Delta_k(U) \subseteq \Lambda_k(U)$.*

Proof. Let $\lambda \in \Delta_k(U)$ be expressed as a convex combination of the eigenvalues for U for each i:

$$\lambda = \sum_{j \in S_i} t_{ij} \lambda_j, \quad t_{ij} \geq 0, \quad \sum_{j \in S_i} t_{ij} = 1.$$

Let $|\phi_i\rangle = \sum_{j \in S_i} \sqrt{t_{ij}} |\psi_j\rangle$. Then $|\phi_1\rangle, \ldots, |\phi_k\rangle$ are orthonormal (orthogonal because the S_i are disjoint). Let P be the orthogonal rank-k projection onto $\mathcal{V} = \mathrm{span}\{|\phi_1\rangle, \ldots, |\phi_k\rangle\}$. Then for each $|\psi\rangle \in \mathcal{V}$, the vector $(U - \lambda I)|\psi\rangle$ is orthogonal to $\mathcal{V}$. Indeed, if $i \neq i'$ then $(U - \lambda I)|\phi_i\rangle \in \mathrm{span}\{|\psi_j\rangle : j \in S_i\}$, which in turn is orthogonal to $|\phi_{i'}\rangle$ because $|\phi_{i'}\rangle \in \mathrm{span}\{|\psi_j\rangle : j \in S_{i'}\}$ and $S_i \cap S_{i'} = \emptyset$. On the other hand, $\langle (U - \lambda I)\phi_i | \phi_i \rangle = \langle U\phi_i | \phi_i \rangle - \lambda$ and

$$\langle U\phi_i | \phi_i \rangle = \langle \sum_{j \in S_i} \lambda_j \sqrt{t_{ij}} \psi_j | \sum_{j \in S_i} \sqrt{t_{ij}} \psi_j \rangle = \sum_{j \in S_i} t_{ij} \lambda_j = \lambda.$$

Hence $PUP = \lambda P$; that is, λ belongs to $\Lambda_k(U)$. ∎

The converse inclusion holds in a wide variety of cases.

Theorem 5.2. *If $N \geq 3k$, then $\Omega_k(U) = \Delta_k(U)$. Hence, Conj($N, k$) holds whenever $N \geq 3k$.*

Proof. Let $T(a, b, c)$ denote the "eigentriangle" $\mathrm{conv}(\{\lambda_a, \lambda_b, \lambda_c\})$. We shall say $T(a, b, c)$ and $T(a', b', c')$ are "disjoint" if the index sets $\{a, b, c\}$ and $\{a', b', c'\}$ are disjoint; of course, the eigentriangles themselves may very well overlap. We show that $N \geq 3k$ implies that every $\lambda \in \Omega_k(U)$ lies in k (pairwise) disjoint eigentriangles, and so $\lambda \in \Delta_k(U) \subseteq \Lambda_k(U)$

by Lemma 5.1. This is clear for $k = 1$ since any convex polygon (here $\Omega_1(U)$, that is $\mathrm{conv}(\{\lambda_j : j = 1, 2, \ldots, N\})$ is (in many ways) a union of triangles formed from the vertices.

For $k > 1$ we proceed by induction. The wedge $W = D(1, k+1, U) \cap D(N-k+1, 1, U))$ contains $\Omega_k(U)$. Consider the eigentriangles

$$T(1, k+1, 2k+1), \quad \ldots \quad , T(1, k+(N-3k+1), 2k+(N-3k+1));$$

note that (because $N \geq 3k$) $2k+1 \leq N-k+1$ and that the last eigentriangle in this list is $T(1, N-2k+1, N-k+1)$. Thus the union of these overlapping eigentriangles covers the part of W that contains $\Omega_k(U)$, and hence covers $\Omega_k(U)$ itself. Given any $\lambda \in \Omega_k(U)$, choose one of the eigentriangles from this list that contains λ. Let the chosen eigentriangle be $T(1, b, c)$; note that $b \geq k+1$, $c = b+k$, and $c \leq N-k+1$. We claim that λ is also in $\Omega_{k-1}(W)$, where $\mathrm{spec}(W) = \{\lambda_j : j \neq 1, b, c\}$. Assuming this claim is correct for the moment, we see that the inductive step is achieved, since $|\,\mathrm{spec}(W)| = N-3 \geq 3(k-1)$ so that λ lies in $k-1$ disjoint eigentriangles drawn from $\mathrm{spec}(W)$ as well as in $T(1, b, c)$, and hence that $\lambda \in \Delta_k(U)$.

To verify the claim keep Lemma 4.1 in mind and note that the sets $D(j, j+k-1, W)$ that intersect to form $\Omega_{k-1}(W)$ strictly contain one of the $D(i, i+k, U)$ unless the arc omitted from $D(j, j+k-1, W)$ includes one of $\lambda_1, \lambda_b, \lambda_c$, in which case $D(j, j+k-1, W)$ coincides with one of the $D(i, i+k, U)$. The key point is that the arc cannot contain *more* than one of $\lambda_1, \lambda_b, \lambda_c$ since these are separated by at least $k-1$ points in $\mathrm{spec}(W)$. Thus, in fact, $\Omega_{k-1}(W) \supseteq \Omega_k(U)$, and this completes the proof. ■

Let us discuss the construction of codes. The cases in which k divides N are perhaps the simplest cases in which codes can be explicitly constructed, as described in the proof of Corollary 4.4. The proof of $\mathrm{Conj}(N, k)$ for $N \geq 3k$ given in Theorem 5.2 is also constructive in the sense that the $\lambda \in \Lambda_k(U)$ and the corresponding projections P may be found explicitly by an algorithm based on the proof. We state this in terms of the general binary unitary channel error correction problem.

Let U be a unitary on $\mathbb{C}^N$ and let k be a positive integer such that $N \geq 3k$. Then by Theorem 4.7 (4) and Theorem 5.2, we have $\Delta_k(U) = \Lambda_k(U) = \Omega_k(U)$ and this set is nonempty. Following the proof of Theorem 5.2 (and recalling our earlier notation), a k-dimensional correctable code for any channel of the form $\mathcal{E} = \{\sqrt{p}\, I, \sqrt{1-p}\, U\}$ can be constructed by:

(i) Compute $\Omega_k(U)$ from the eigenvalues $\{\lambda_1, \dots, \lambda_N\}$ of U. This can be done by using Lemma 4.1 in general, or by simpler means in special cases, such as that of Corollary 4.4.

(ii) Choose $\lambda \in \Omega_k(U)$. By Theorem 5.2, we can find k-eigentriangles $T(a_j, b_j, c_j)$, $j = 1, \dots, k$, such that each contains λ and there are no repeats in the set $\{a_j, b_j, c_j\}_{j=1}^k \subseteq \{1, \dots, N\}$.

(iii) As λ belongs to each of the convex hulls $\mathrm{conv}\{\lambda_{a_j}, \lambda_{b_j}, \lambda_{c_j}\}$, for $j = 1, \dots, k$, we can compute $t_{1j}, t_{2j}, t_{3j} \geq 0$, $\sum_{i=1}^3 t_{ij} = 1$ such that $\lambda = t_{1j}\lambda_{a_j} + t_{2j}\lambda_{b_j} + t_{3j}\lambda_{c_j}$.

(iv) For $j = 1, \dots, k$, put $s_{ij} = \sqrt{t_{ij}}$ and define (orthonormal) states $|\phi_j\rangle = s_{1j}|\psi_{a_j}\rangle + s_{2j}|\psi_{b_j}\rangle + s_{3j}|\psi_{c_j}\rangle$. Let $P = \sum_{j=1}^k |\phi_j\rangle\langle\phi_j|$. Then $P\mathbb{C}^N = \mathrm{span}\{|\phi_1\rangle, \dots, |\phi_N\rangle\}$ is a k-dimensional correctable code for $\mathcal{E}$.

Remark 5.3. *It is instructive to rephrase this construction in terms of the number of qubits, assuming that the entire system has dimension $N = 2^n$. The codes constructed above work for $k = [N/3]$, which for $n \geq 2$ is not smaller than $N/4 = 2^{n-2}$. Therefore these error-correcting codes support $n - 2$ logical qubits. Our construction shows that such codes are parametrized by the complex numbers $\lambda \in \Lambda_k(U)$, which is a nonempty set explicitly determined by the eigenvalues for U as we have shown. This result is optimal for $n \geq 3$ in the sense that for a generic unitary one cannot obtain a code preserving $n - 1$ qubits. This observation follows from the fact that $\Lambda_{N/2}(U)$ is not empty only in a very specific situation; if the $N/2$ lines joining opposite (with respect to the ordering number of the phase) eigenvalues of U cross in a single point.*

This result considered in the context of qudits (which denote d-level systems) has the following implication: If $N = d^n$ and $d \geq 3$ then $k = [N/3] \geq N/d$, so the constructed code supports $d - 1$ qudits.

6. Non-Constructive Verifications of the Conjecture

In this section we derive a non-constructive verification of $\mathrm{Conj}(N, k)$ in the case $N = 5$, $k = 2$, and then we extend the proof to a variety of cases. For convenience we consider only U with distinct eigenvalues.

To move beyond the limitations of $\Delta_k(U)$ we introduce $\Sigma_k(U)$ as the set of all $\lambda \in \mathrm{conv}(\mathrm{spec}(U))$ such that for some single convex combination

$$\lambda = \sum_j t_j\lambda_j, \quad t_j \geq 0, \quad \sum_j t_j = 1$$

we have $\alpha_{ij} \in \mathbb{C}$ ($i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, N$) with the following properties: $|\alpha_{ij}| = \sqrt{t_j}$ for all i and j, and whenever $i \neq i'$ we have both $\sum_j \alpha_{ij}\overline{\alpha_{i'j}} = 0$ and $\sum_j \lambda_j \alpha_{ij}\overline{\alpha_{i'j}} = 0$. The symbol Σ in $\Sigma_k(U)$ is chosen to recall that we use a *single* convex combination to represent each λ.

It is easy to see that $\Sigma_k(U) \subseteq \Lambda_k(U)$: consider any $\lambda \in \Sigma_k(U)$ and (using again the above notation) let $|\phi_i\rangle = \sum_j \alpha_{ij}|\psi_j\rangle$. The conditions on the α_{ij} directly imply that the $|\phi_i\rangle$ are orthonormal, that $\langle U\phi_i|\phi_i\rangle = \lambda$ for each i, and that $\langle U\phi_i|\phi_{i'}\rangle = 0$ whenever $i \neq i'$. Let P be the rank-k orthogonal projection onto the subspace $\mathcal{V} = \operatorname{span}\{|\phi_1\rangle, \ldots, |\phi_k\rangle\}$. Clearly $(U - \lambda I)\mathcal{V}$ is orthogonal to $\mathcal{V}$, so that $PUP = \lambda P$ and $\lambda \in \Lambda_k(U)$.

Let σ denote spec(U) considered as a vector $(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{C}^N$. In terms of σ the following lemma provides a recipe for making elements of $\Sigma_2(U)$.

Lemma 6.1. *Given $p \in \mathbb{C}^N$ such that $\vec{0} \neq p \perp \{\vec{1}, \sigma, \overline{\sigma}\}$, set $s(p) = \sum_j |p_j|$ and*

$$f(p) = \sum_j \frac{|p_j|}{s(p)}\lambda_j.$$

Then $f(p) \in \Sigma_2(U)$.

Proof. Let $\alpha_{1j} = \sqrt{|p_j|/s(p)}$ and $\alpha_{2j} = \alpha_{1j}\overline{p_j}/|p_j|$ (if $p_j = 0$, let $\alpha_{2j} = \alpha_{1j}$ $(= 0)$). With $\lambda = f(p)$ and $t_j = |p_j|/s(p)$ we easily verify the requirements for the α_{ij}. For example, $\sum_j \lambda_j \alpha_{1j}\overline{\alpha_{2j}} = (\sum_j \lambda_j p_j)/s(p) = (p, \overline{\sigma})/s(p) = 0$. ■

Corollary 6.2. *For any $k \geq 2$ we have $\Delta_k(U) \subseteq \Sigma_2(U)$.*

Proof. Consider $\lambda \in \Delta_k(U)$. Using again the notation from the proof of Lemma 5.1, let $t_{1*} = (t_{11}, t_{12}, \ldots, t_{1N})$ with the understanding that $t_{1j} = 0$ if $j \notin S_1$; similarly define t_{2*}. Let $p = t_{1*} - t_{2*}$. Then $p \neq \vec{0}$ since the supports of t_{1*} and t_{2*} are disjoint; $(p, \vec{1}) = \sum_j t_{1j} - \sum_j t_{2j} = 1 - 1 = 0$; $(p, \sigma) = \sum_j t_{1j}\overline{\lambda_j} - \sum_j t_{2j}\overline{\lambda_j} = \overline{\lambda} - \overline{\lambda} = 0$; similarly p is perpendicular to $\overline{\sigma}$. Thus $f(p) \in \Sigma_2(U)$. Finally, since the supports of t_{1*} and t_{2*} are disjoint, $f(p) = (\sum_j t_{1j}\lambda_j + \sum_j t_{2j}\lambda_j)/2 = (\lambda + \lambda)/2 = \lambda$. ■

Remark 6.3. *Although Corollary 6.2 is sufficient for our present needs, it should be noted that in fact $\Delta_k(U) \subseteq \Sigma_k(U)$ for any k. Indeed, for*

$\lambda \in \Delta_k(U)$ there are $t_{ij} \geq 0$ ($i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, N$) such that $\sum_j t_{ij} = 1$, $\sum_j t_{ij}\lambda_j = \lambda$, and for each j at most one of the t_{ij} is nonzero. Thus, setting $s_{ij} = \sqrt{t_{ij}}$ we have a $k \times N$ matrix S such that $S\,\mathrm{diag}(\lambda_j)S^* = \lambda I_k$ and $SS^* = I_k$. Let F be any $k \times k$ unitary matrix with $|f_{ij}| = 1/\sqrt{k}$ (for all i, j); for example, F could be the "finite Fourier transform", where $f_{ij} = \omega^{ij}/\sqrt{k}$ with ω a primitive k–th root of unity. Setting $R = FS$ we have $R\,\mathrm{diag}(\lambda_j)R^* = F\lambda I_k F^* = \lambda I_k$, and $RR^* = I_k$. Thus we may verify that $\lambda \in \Sigma_k(U)$ by considering $\alpha_{ij} = r_{ij}$, since (for each j) $|\alpha_{ij}|$ is then independent of i; in fact, for the given j just one $s_{mj} \neq 0$ so that $|\alpha_{ij}| = |f_{im}s_{mj}| = |s_{mj}|/\sqrt{k}$.

It is clear that when $N = 5$ only the boundary $\partial\Omega_2(U)$ is captured by $\Delta_2(U)$; each point on an edge of the pentagon $\Omega_2(U)$ belongs both to a line segment $\mathrm{conv}(\{\lambda_i, \lambda_{i+2}\})$ and to the complementary eigentriangle $T(i-1, i+1, i+3)$. To capture the interior of $\Omega_2(U)$ we turn to $\Sigma_2(U)$. The topological techniques used in the proof of the following result are interesting mathematically but make the construction of specific projections P corresponding to $\lambda \in \Lambda_2(U)$ more difficult than we saw with techniques based on $\Delta_k(U)$.

Theorem 6.4. *If $N = 5$, then $\Omega_2(U) = \Sigma_2(U)$. Thus, Conj(5,2) is correct.*

Proof. By Corollary 6.2 we know that $\partial\Omega_2(U) = \Delta_2(U) \subseteq \Sigma_2(U)$; to capture the interior we first elaborate the ideas in the proof of that corollary. Let a_i be the vertex of $\Omega_2(U)$ at the point of intersection between the line segment $[\lambda_i, \lambda_{i+2}]$ (in other words, $\mathrm{conv}(\{\lambda_i, \lambda_{i+2}\})$) and the line segment $[\lambda_{i+1}, \lambda_{i+3}]$. Each $a \in [a_i, a_{i+1}]$ also lies in $[\lambda_{i+1}, \lambda_{i+3}]$ and so has a unique representation as a convex combination $\sum_j t_{ij}(a)\lambda_j$ with $t_{ij}(a) = 0$ when $j \notin \{i+1, i+3\}$. Likewise $a \in [a_i, a_{i+1}]$ also lies in the eigentriangle $T(i, i+2, i+4)$ and so has a unique representation as a convex combination $\sum_j s_{ij}(a)\lambda_j$ with $s_{ij}(a) = 0$ when $j \in \{i+1, i+3\}$.

For each $a \in [a_i, a_{i+1}]$, let $p_i(a) = t_{i*}(a) - s_{i*}(a)$; as in the proof of the last corollary, we see that $p_i(a) \in X$ where $X = \{\vec{1}, \sigma, \overline{\sigma}\}^{\perp} \backslash \{\vec{0}\}$ and that $a = f(p_i(a))$. Note that $X \equiv \mathbb{C}^2 \setminus \{\vec{0}\} \equiv \mathbb{R}^4 \setminus \{\vec{0}\}$, so that X is simply connected. Moreover, each p_i is continuous on $[a_i, a_{i+1}]$. Because of the uniqueness of the representations as convex combinations, $t_{i*}(a_{i+1}) = s_{(i+1)*}(a_{i+1})$ and $s_{i*}(a_{i+1}) = t_{(i+1)*}(a_{i+1})$. Thus $p_i(a_{i+1}) = -p_{i+1}(a_{i+1})$. Let $\gamma_0 : [0, 10] \to \partial\Omega_2(U)$ be a (continuous) path traversing $\partial\Omega_2(U)$ *twice* in the counterclockwise direction, beginning and ending at a_1 and such that $\gamma_0([j, j+1]) = [a_{j+1}, a_{j+2}]$ ($j = 0, 1, \ldots, 9$) with the understanding that the a_i are numbered modulo 5 ($= N$). For $t \in$

$[j, j+1]$ let $\Gamma_0(t) = (-1)^j p_{j+1}(\gamma_0(t))$. The alternating signs ensure that Γ_0 is continuous (even at the integers), and the double circuit ensures that $\Gamma_0(0) = \Gamma_0(10)$; in other words, that Γ_0 is a *loop* in X. Furthermore $f(\Gamma_0(t)) = f(\pm p_{j+1}(\gamma_0(t))) = \gamma_0(t)$.

Thus, given any λ in the interior of $\Omega_2(U)$, the winding number of $f \circ \Gamma_0$ relative to λ is 2; that is, $w_\lambda(f \circ \Gamma_0) = 2$. Since X is simply connected, the loop Γ_0 is part of a continuous family of loops Γ_s $(0 \le s \le 1)$ in X such that Γ_1 is the constant loop at some $p_* \in X$. Suppose λ does not lie on any of the loops $f \circ \Gamma_s$; then $w_\lambda(f \circ \Gamma_s) = 2$ for all s, a contradiction, since $w_\lambda(f \circ \Gamma_1) = 0$ ($f \circ \Gamma_1$ is the constant loop at $f(p_*)$). Thus, for some s, t, $f(\Gamma_s(t)) = \lambda$. Since $\Gamma_s(t) \in X$, Lemma 6.1 implies that $\lambda \in \Sigma_2(U)$. ■

This result has a number of consequences, as follows.

Corollary 6.5. *For every natural number m, Conj($5m, 2m$) is valid.*

Proof. Let $S_j = \{j, j+m, j+2m, j+3m, j+4m\}$ $(j = 1, 2, \ldots, m)$. The S_j partition $\{1, 2, \ldots, 5m\}$ into m disjoint subsets. In view of Lemma 4.1, we have

$$\Omega_{2m}(U) = \bigcap_{i=1}^{5m} D(i, i+2m, U) =$$

$$\bigcap_{j=1}^{m}\Big(\bigcap_{i \in S_j} D(i, i+2m, U)\Big) = \bigcap_{j=1}^{m}\Big(\bigcap_{i=1}^{5} D(i, i+2, U_j)\Big),$$

where U_j is the unitary with spectrum $\operatorname{spec}(U_j) = \{\lambda_i : i \in S_j\}$. Thus $\Omega_{2m}(U) = \bigcap_{j=1}^m \Omega_2(U_j)$ and this is $\bigcap_{j=1}^m \Lambda_2(U_j)$ by Theorem 6.4 (since $|\operatorname{spec}(U_j)| = 5$).

Consider any $\lambda \in \Omega_{2m}(U)$; for each $j = 1, 2, \ldots, m$ this $\lambda \in \Lambda_2(U_j)$ and we have a 2–dimensional subspace $\mathcal{V}_j$ of $\operatorname{span}(\{|\psi_i\rangle : i \in S_j\})$ such that $(U - \lambda I)\mathcal{V}_j \perp \mathcal{V}_j$. Now $\mathcal{V}_j$ and $(U - \lambda I)\mathcal{V}_j$ are subspaces of the mutually orthogonal

$$\operatorname{span}\{|\psi_i\rangle : i \in S_j\} \;\; (j = 1, 2, \ldots, m).$$

Thus $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \cdots + \mathcal{V}_m$ is a $2m$–dimensional subspace such that $(U - \lambda I)\mathcal{V} \perp \mathcal{V}$, so that $\lambda \in \Lambda_{2m}(U)$. ■

Remark 6.6. *Although Corollary 6.4 ensures that Conj(15,6) is correct, for instance, it may not be useful in applications because it can happen that $\Omega_6(U) = \emptyset$ when $N = 15$ ($15 = 3{\cdot}6{-}3$; recall Theorem 4.6). Moreover, $\Omega_7(V)$ may be empty in dimension 16 ($16 \le 3 \cdot 7 - 3$) so that* *we cannot use Proposition 4.5 to support the "induction" Conj(15,6) $\Longrightarrow$ Conj(16,6). Thus Conj(16,6) remains undecided at the moment.*

Corollary 6.7. *Conj($3k-1,k$) holds for all $k \geq 1$.*

Proof. The case $k = 1$ is trivial (for all N). The case $k = 2, N = 5$ was proved in Theorem 6.4. Based on this we can make an induction on k somewhat similar to that used in Theorem 5.2. For $k+1 > 2$ consider U with $|\operatorname{spec}(U)| = N = 3(k+1) - 1$. Let W_1 denote the wedge

$$W_1 = D(1, 1+(k+1), U) \cap D(1+(2k+1), 1+(2k+1)+(k+1), U).$$

Note that $\Omega_{k+1}(U) \subseteq W_1$ and in fact $\Omega_{k+1}(U)$ is contained in the eigentriangle $T_1 = T(1, 1+(k+1), 1+(2k+1))$, since $(1+(2k+1)) - (1+(k+1)) = k < k+1$ and $1+(2k+1)+(k+1) = N+1 \equiv 1$. Let U_1 be the unitary with spectrum

$$\operatorname{spec}(U_1) = \operatorname{spec}(U) \setminus \{\lambda_1, \lambda_{1+(k+1)}, \lambda_{1+(2k+1)}\}.$$

Then $\Omega_k(U_1) \supseteq \Omega_{k+1}(U)) \cap D(1)$, where $D(1) = D(1+k, 1+(2k+2), U)$. Thus any $\lambda \in \Omega_{k+1}(U) \cap D(1)$ is in $\Omega_k(U_1)$ as well as in T_1. Since $|\operatorname{spec}(U_1)| = 3k-1$ the inductive hypothesis ensures that $\lambda \in \Lambda_k(U_1)$ and there is a k–dimensional subspace $\mathcal{V}_1$ of span($\{|\psi_i\rangle : \lambda_i \in \operatorname{spec}(U_1)\}$) such that $(U - \lambda I)\mathcal{V}_1 \perp \mathcal{V}_1$. Since $\lambda \in T_1$ we also have a 1–dimensional subspace $\mathcal{V}_1'$ of span$\{|\psi_1\rangle, |\psi_{1+(k+1)}\rangle, |\psi_{1+(2k+1)}\rangle\}$ such that $(U - \lambda I)\mathcal{V}_1' \perp \mathcal{V}_1'$. The (orthogonal) sum $\mathcal{V}_1 + \mathcal{V}_1'$ shows that $\lambda \in \Lambda_{k+1}(U)$.

Similarly we have $D(k+3) = D(k+3+k, k+3+(2k+2), U)$ such that $\lambda \in \Omega_{k+1}(U) \cap D(k+3)$ implies $\lambda \in \Lambda_{k+1}(U)$. Finally, $D(1) \cup D(k+3) = \mathbb{D}$, since $1+(2k+2) = k+3+k$ and $k+3+(2k+2) < N+1+k$ (if and only if $k+1 > 2$). ■

Corollary 6.8. *For $N = 7$, we at least have $\partial\Omega_3(U) \subseteq \Lambda_3(U)$.*

Proof. Along the lines of the proofs above, we need only show that $\lambda \in \partial\Omega_3(U)$ implies $\lambda \in \Omega_1(U') \cap \Omega_2(U'')$, where the spectra σ', σ'' of U', U'' partition the spectrum σ of U, $|\sigma'| = 2$, and $|\sigma''| = 5$. Then $\lambda \in \Lambda_1(U')$ and, using Theorem 6.3 we also have $\lambda \in \Lambda_2(U'')$. To complete the argument note that, if λ belongs to one of the line segments forming a side of $\partial\Omega_3(U)$, then $\lambda \in [\lambda_i, \lambda_{i+3}]$ for some i and we set $\sigma' = \{\lambda_i, \lambda_{i+3}\}$. Since λ_{i+1} and λ_{i+2} are the only points of $\sigma'' = \sigma \setminus \sigma'$ in the counterclockwise arc from λ_i to λ_{i+3}, we must also have $\lambda \in \Omega_2(U'')$. ■

Remark 6.9. *If there were some sort of* a priori *convexity result for the sets $\Lambda_k(T)$ – along the lines of the Hausdorff–Toeplitz Theorem for the classical numerical range (= $\Lambda_1(T)$) – many of our arguments could be simplified and extended. For example, from Corollary 6.6 we could derive Conj(7,3). If Conj(N, k) is true in general then we also have convexity of all $\Lambda_k(T)$ for all normal T, but we do not know of any independent argument for such convexity. In fact, Conj(7,3) seems to be the "smallest" case that presently remains unsettled.*

7. Outlook

As discussed above, the full Conjecture A remains open. Here we have constructively verified the conjecture in a wide variety of cases, and, curiously, non-constructively in other cases. An overarching conceptual proof covering all cases would be of great interest. A possible avenue to such a result could come through a general convexity theorem for higher-rank numerical ranges, independent of normality or unitarity, though the present work suggests that establishing such a result would be a delicate matter. Progress in this direction is contained in the recent work [**4**].

To apply the compression approach [**5**] to broader classes of noise maps (in particular to randomized unitary channels with more than two unitary errors) as a means to construct ideal correctable codes, a better understanding is required of joint solutions to the family of equations given by Eqs. (2). Furthermore, we have focussed on the generic case of non-degenerate spectrum to streamline the presentation. But many naturally arising physical examples include degenerate spectra. There are extra technical issues to overcome in such cases, but we expect our results can be extended to the case of degenerate spectra.

It would be interesting to consider possible infinite-dimensional extensions of these results. We have also not studied here possible implications of the compression approach to more subtle subsystem codes [**11, 12**]. Nor have we considered possible applications to approximate error-correction [**2, 7, 14, 16**], and specifically to noise maps that have two unitary errors which occur with high probability.

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